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ON THE THEOREMS FOR LOCAL VOLUME AVERAGING OF MULTIPHASE SYSTEMS

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Abstract—Proofs of the theorems of local volume averaging which relate averages of derivatives to derivatives of averages are presented. A distribution function whose derivatives are proportional to the Dirac delta function is used in the development and the proofs demonstrated are intended to be simpler than those found in the literature.

INTRODUCTION

The technique of local volume averaging of continuum equations of motion and transport has been shown by various authors (e.g. Whitaker 1967; Slattery 1967; Bear 1972; Gray & O'Neill 1976) to be of value in obtaining equations applicable to multiphase systems. Whitaker (1966) showed that local surface averaging could be used to develop Darcy's law. This development, however, is severely complicated by notational conventions used to ensure consistency of the averaging process. Whitaker (1967) and Slattery (1967) were able to overcome the conceptual and notational difficulties of area averaging by volume averaging. For this process to lead to averaged equations which are useful, a theorem must be applied which relates the average of a spatial derivative of a function to the spatial derivative of the average of the function. Slattery (1967) and Whitaker (1969) developed this theorem by analogy with the transport theorem. Additional arguments concerning the derivative of an average and its physical significance were presented to complete the proof. Bachmat (1972) has developed a proof of the averaging theorem for spatial derivatives as well as for the time derivative. All the above proofs of the averaged theorems are conceptually very difficult.

In this paper alternative proofs of the averaging theorems are presented. These proofs are intended to offer simpler derivations than reported previously and to clarify misconceptions that may arise concerning the assumptions and utility of the local averaging technique.

DISTRIBUTION FUNCTION

The purpose of subsequent discussion is to obtain averaging theorems appropriate for local averaging of flow and transport equations in porous media. To this end a function γ_α is defined which has a value of unity in the α-phase but is zero in all other phases which will collectively be referred to as the β-phase. For example, if equations for the fluid phase in saturated flow in a groundwater aquifer are to be obtained, that phase would be referred to as the α-phase while the solid matrix would form the β-phase. In unsaturated flow, α might refer to the gas phase while the liquid and solid phases would together comprise the β-phase. By allowing α to denote any of the phases present, averaged equations may be developed for that phase.

A building block for the function γ_α is the unit step function which in one dimension is defined by

$$H(x_1 - a) = \begin{cases} 0, & x_1 < a \\ 1, & x_1 \geq a. \end{cases} \quad [1]$$

The derivative of this function is given by

$$\frac{\mathrm{d}H(x_1 - a)}{\mathrm{d}x_1} = \delta(x_1 - a) \qquad [2]$$

where $\delta(x_1 - a)$ is the Dirac delta function (Lighthill 1958).

Now consider the function $\gamma_\alpha(x_1)$ in figure 1a which is the sum of the step functions

$$\gamma_\alpha(x_1) = H(x_1 - a_0) - H(x_1 - a_1) + H(x_1 - a_2) - H(x_1 - a_3) + H(x_1 - a_4). \qquad [3]$$

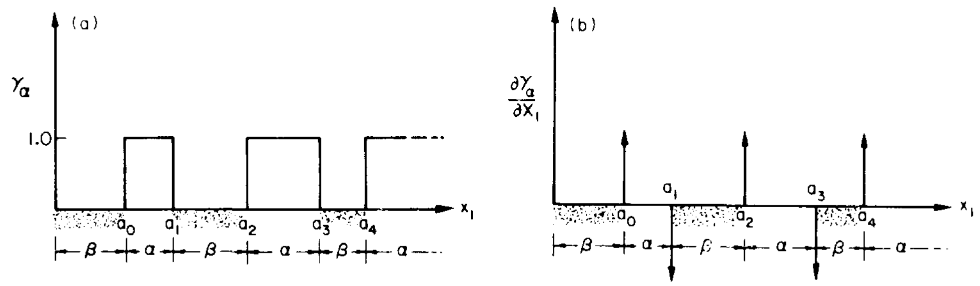

Figure 1. One-dimensional distribution function γ_α (defined to be unity in the α-phase and zero in the β-phase) and its derivatives.

The derivative of this function with respect to x_1 is a sum of Dirac delta functions

$$\frac{\mathrm{d}\gamma_\alpha(x_1)}{\mathrm{d}x_1} = \delta(x_1 - a_0) - \delta(x_1 - a_1) + \delta(x_1 - a_2) - \delta(x_1 - a_3) + \delta(x_1 - a_4) \qquad [4]$$

and is plotted in figure 1b. If a unit normal vector $\mathbf{n}_\alpha$ is defined which points outward from the α-phase to the β-phase at the α–β interface, then [4] can be rewritten

$$\frac{\mathrm{d}\gamma_\alpha(x_1)}{\mathrm{d}x_1} = -\sum_{k=0}^{4} \mathbf{n}_\alpha \cdot \mathbf{i}\delta(x_1 - a_k) \qquad [5]$$

where $\mathbf{i}$ is a unit vector in the positive x_1 direction.

A two-dimensional distribution, $\gamma_\alpha(x_1, x_2)$ is demonstrated in figure 2a. In this instance

$$\frac{\partial\gamma_\alpha(x_1, x_2)}{\partial x_1} = -\mathbf{n}_\alpha \cdot \mathbf{i}\delta(\mathbf{x} - \mathbf{x}_{\alpha\beta}), \qquad [6a]$$

$$\frac{\partial\gamma_\alpha(x_1, x_2)}{\partial x_2} = -\mathbf{n}_\alpha \cdot \mathbf{j}\delta(\mathbf{x} - \mathbf{x}_{\alpha\beta}) \qquad [6b]$$

where $\mathbf{x}$ is a position vector, and $\mathbf{x}_{\alpha\beta}$ is the position vector of the α–β interface.

Plots of $(\partial\gamma_\alpha(x_1, x_2)/\partial x_1)$ and $(\partial\gamma_\alpha(x_1, x_2)/\partial x_2)$ obtained with reference to figure 2a appear as figures 2b and c. Equations [6] can be re-expressed in vector notation as

$$\nabla^*\gamma(x_1, x_2) = -\mathbf{n}_\alpha\delta(\mathbf{x} - \mathbf{x}_{\alpha\beta}) \qquad [7]$$

where

$$\nabla^* = \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}.$$

By analogy with [7], the gradient of a three-dimensional distribution function $\gamma_\alpha(x_1, x_2, x_3)$ [which also will be written as $\gamma_\alpha(\mathbf{x})$] is

$$\nabla_x \gamma_\alpha(\mathbf{x}) = -\mathbf{n}_\alpha \delta(\mathbf{x} - \mathbf{x}_{\alpha\beta}) \qquad [8]$$

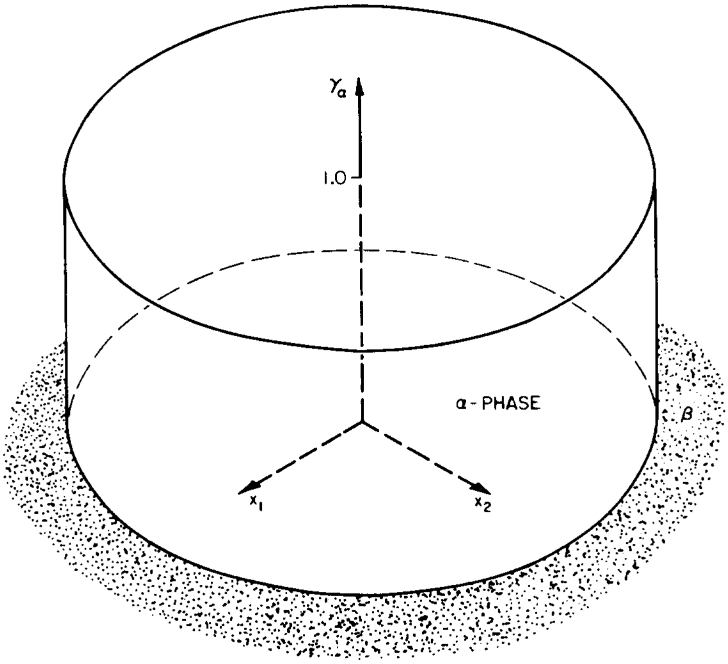

Figure 2(a)

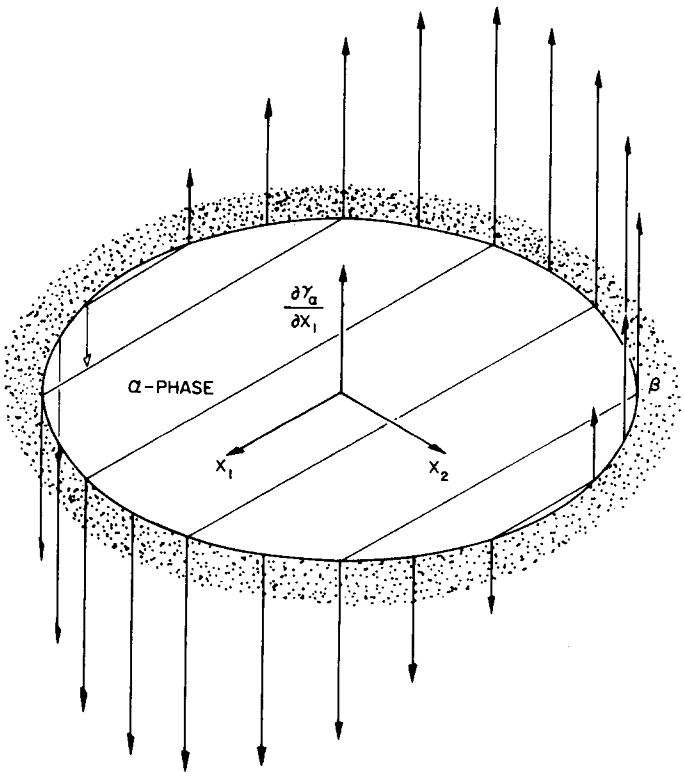

Figure 2(b)

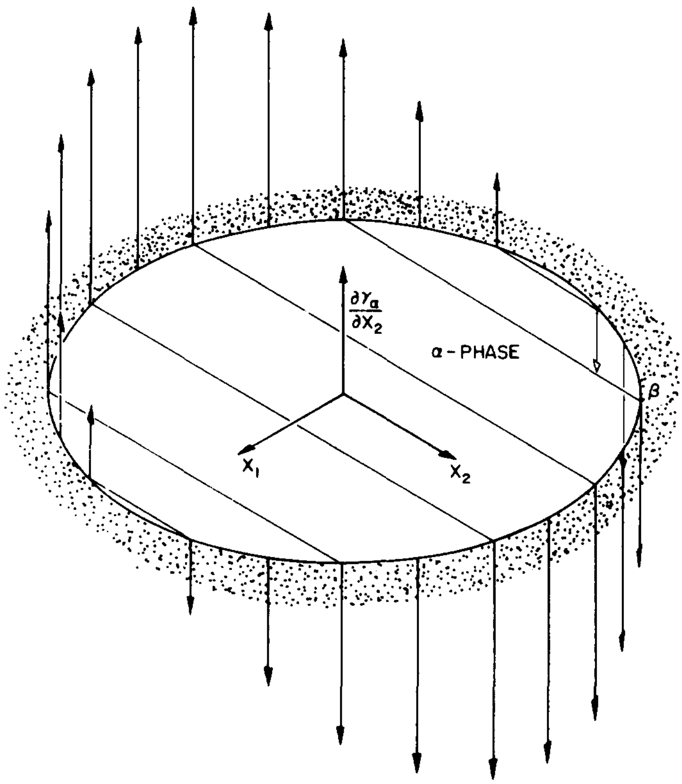

Figure 2(c)

Figure 2. Two-dimensional distribution function γ_α (defined to be unity in the α-phase and zero in the β-phase) and its derivatives.

where

$$\nabla_x = \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}.$$

It should be noted that for a medium in which the α-phase is deforming, γ_α will be a function of time as well as space.

DEFINITIONS FOR AVERAGING

When applying averaging techniques to continuum equations for flow in porous media, it is necessary to select an averaging volume that will result in meaningful averages. Whitaker (1969) has shown that this condition can be met when the characteristic length of the averaging volume is much greater than the pore diameter in the medium but much less than the characteristic length of the medium. Additionally the shape, size, and orientation of the averaging volume will be required to be independent of space and time.

For purposes of averaging, it is convenient to define a local co-ordinate system, ξ_1, ξ_2, ξ_3, which has axes parallel with the x_1, x_2, x_3 system but whose origin is located at position $\mathbf{x}$ (figure 3). The location of the averaging volume with respect to the $\boldsymbol{\xi}$ co-ordinate system is independent of $\mathbf{x}$. For example, the averaging volume may be defined such that its centroid always coincides with the origin of the $\boldsymbol{\xi}$ system.

The phase average, $\langle\psi_\alpha\rangle$, of some property ψ is now defined by

$$\langle\psi_\alpha\rangle(\mathbf{x}, t) = \frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi}, t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)\,\mathrm{d}V_\xi \qquad [9]$$

where the volume of integration, $V = V_\alpha + V_\beta$, is independent of space and time. However V_α

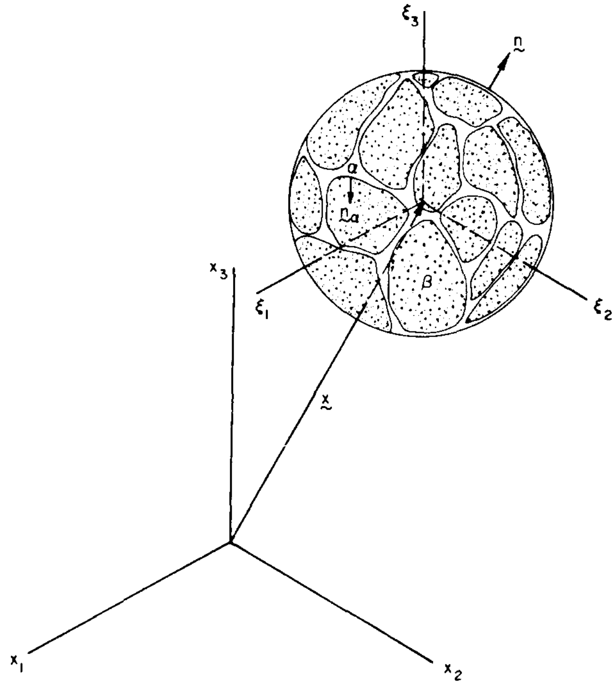

Figure 3. Local averaging volume V containing both α- and β-phases.

and V_β will depend on **x** and will also depend on t if the medium is deformable. Physically, the phase average is a property of the α-phase only averaged over the entire volume occupied by the α- and β-phases in the averaging volume (e.g. specific discharge is the phase average of the fluid velocity). Because γ_α is zero in the β-phase, [9] can alternatively be written

$$\langle\psi_\alpha\rangle(\mathbf{x}, t) = \frac{1}{V}\int_{V_\alpha(\mathbf{x}, t)} \psi(\mathbf{x}+\boldsymbol{\xi}, t)\,\mathrm{d}V_\xi \qquad [10]$$

but in this instance, the limits of integration depend on spatial location and on time if the medium deforms.

The intrinsic phase average, $\langle\psi_\alpha\rangle^\alpha$, of some property ψ is given by

$$\langle\psi_\alpha\rangle^\alpha(\mathbf{x}, t) = \frac{1}{V_\alpha(\mathbf{x}, t)}\int_{V_\alpha(\mathbf{x}, t)} \psi(\mathbf{x}+\boldsymbol{\xi}, t)\,\mathrm{d}V_\xi. \qquad [11]$$

This type of average describes a property of the α-phase averaged over that phase only (e.g. the fluid velocity obtained by averaging the point fluid velocities over the volume occupied by the fluid is an intrinsic phase average). Comparison of [10] and [11] indicates that

$$\langle\psi_\alpha\rangle(\mathbf{x}, t) = \epsilon_\alpha(\mathbf{x}, t)\langle\psi_\alpha\rangle^\alpha(\mathbf{x}, t) \qquad [12]$$

where

$$\epsilon_\alpha(\mathbf{x}, t) = V_\alpha(\mathbf{x}, t)/V = \frac{1}{V}\int_V \gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)\,\mathrm{d}V_\xi. \qquad [13]$$

Thus ϵ_α is the porosity or in the fraction of the medium occupied by the α-phase.

Before proceeding to the averaging theorems, we shall examine a useful identity involving the gradient operator. For notational convenience the following conventions are adopted:

∇_x refers to the gradient taken with respect to $\mathbf{x}$ coordinates holding ξ_1, ξ_2, and ξ_3 constant;
∇_ξ refers to the gradient taken with respect to $\boldsymbol{\xi}$ coordinates holding x_1, x_2, and x_3 constant; and
∇ refers to either ∇_x or ∇_ξ.

These conventions will be used through the remainder of the paper. If a function is symmetrically dependent on $\mathbf{x}$ and $\boldsymbol{\xi}$ (that is dependent on $\mathbf{x}+\boldsymbol{\xi}$ rather than $\mathbf{x}$ and $\boldsymbol{\xi}$) the gradient of that function in the $\mathbf{x}$ coordinate system is equal to its gradient in the $\boldsymbol{\xi}$ coordinate system. Therefore

$$\nabla_x\psi(\mathbf{x}+\boldsymbol{\xi}) = \nabla_\xi\psi(\mathbf{x}+\boldsymbol{\xi}) = \nabla\psi(\mathbf{x}+\boldsymbol{\xi}), \qquad [14a]$$

$$\nabla_x\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}) = \nabla_\xi\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}) = \nabla\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}). \qquad [14b]$$

AVERAGING THEOREMS

The first theorem of interest relates the average of a gradient to the gradient of an average and was developed by Slattery (1967) and Whitaker (1967). If ψ is continuous within the α-phase, this theorem states that

$$\langle\nabla\psi_\alpha\rangle = \nabla\langle\psi_\alpha\rangle + \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha \mathbf{n}_\alpha \, \mathrm{d}A. \qquad [15]$$

This relation is easily proved using the function γ_α discussed previously. From [9],

$$\langle\nabla\psi_\alpha\rangle = \frac{1}{V}\int_V [\nabla\psi(\mathbf{x}+\boldsymbol{\xi},t)]\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi},t)\,\mathrm{d}V_\xi. \qquad [16]$$

Application of the chain rule to the terms on the R.H.S. of [16] yields

$$\langle\nabla\psi_\alpha\rangle = \frac{1}{V}\int_V \nabla[\psi(\mathbf{x}+\boldsymbol{\xi},t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi},t)]\,\mathrm{d}V_\xi - \frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi},t)[\nabla\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi},t)]\,\mathrm{d}V_\xi. \qquad [17]$$

Substitution of [8] into this equation yields

$$\langle\nabla\psi_\alpha\rangle = \frac{1}{V}\int_V \nabla[\psi(\mathbf{x}+\boldsymbol{\xi},t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi},t)]\,\mathrm{d}V_\xi + \frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi},t)\mathbf{n}_\alpha\delta_\alpha(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}_{\alpha\beta},t)\,\mathrm{d}V_\xi. \qquad [18]$$

The last integral in [18] involves the delta function which is zero everywhere except at the α–β phase interface. The value of an integral whose integrand is a δ-function multiplied by some other quantity is just that quantity evaluated at the singular points of the δ-function. Therefore

$$\frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi},t)\mathbf{n}_\alpha\delta_\alpha(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}_{\alpha\beta},t)\,\mathrm{d}V_\xi = \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha(\mathbf{x}+\boldsymbol{\xi},t)\mathbf{n}_\alpha\,\mathrm{d}A, \qquad [19]$$

and [18] becomes

$$\langle\nabla\psi_\alpha\rangle = \frac{1}{V}\int_V \nabla[\psi(\mathbf{x}+\boldsymbol{\xi},t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi},t)]\,\mathrm{d}V_\xi + \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha\mathbf{n}_\alpha\,\mathrm{d}A. \qquad [20]$$

If ∇ on the R.H.S. of [20] is considered to be ∇_x, it may be removed from the integral sign because the volume of integration has been specified to be independent of $\mathbf{x}$. Thus one obtains

$$\langle\nabla\psi_\alpha\rangle = \nabla_x\left[\frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi},t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi},t)\,\mathrm{d}V_\xi\right] + \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha\mathbf{n}_\alpha\,\mathrm{d}A, \qquad [21]$$

or after making use of [9],

$$\langle\nabla\psi_\alpha\rangle = \nabla_x\langle\psi_\alpha\rangle + \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha \mathbf{n}_\alpha \, dA, \qquad [22]$$

which is the averaging theorem.

If ∇ on the R.H.S. of [20] is considered to be ∇_ξ then the divergence theorem may be applied to the integral and [20] becomes

$$\langle\nabla\psi_\alpha\rangle = \frac{1}{V}\int_\Gamma \mathbf{n}\psi(\mathbf{x}+\boldsymbol{\xi}, t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)\, d\Gamma_\xi + \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha \mathbf{n}_\alpha \, dA, \qquad [23]$$

where Γ is the surface of the averaging volume, and $\mathbf{n}$ is an outward unit normal on Γ. Because γ_α is zero where the surface of the averaging volume is in the β-phase and unity where this surface is in the α-phase, it is only necessary to integrate over Γ_α, the portion of Γ in the α-phase, to evaluate the first integral on the R.H.S. of [23].

Thus [23] becomes

$$\langle\nabla\psi_\alpha\rangle = \frac{1}{V}\int_{\Gamma_\alpha(\mathbf{x},t)} \mathbf{n}\psi_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)\, d\Gamma_\xi + \frac{1}{V}\int_{A_{\alpha\beta}(\mathbf{x},t)} \psi_\alpha \mathbf{n}_\alpha \, dA_\xi. \qquad [24]$$

Equating the R.H.S. of [22] and [24], one obtains

$$\nabla_x\langle\psi_\alpha\rangle = \frac{1}{V}\int_{\Gamma_\alpha(\mathbf{x},t)} \mathbf{n}\psi_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)\, d\Gamma_\xi. \qquad [25]$$

This relation has also been obtained by Slattery (1967) and Whitaker (1969).

The second theorem that will be considered relates the spatial average of a time derivative to the derivative of a spatial average. This theorem was used by Whitaker (1973) who considered it to be the transport theorem associated with a point fixed in space. From [9], the phase average of a time derivative is

$$\left\langle\frac{\partial\psi_\alpha}{\partial t}\right\rangle = \frac{1}{V}\int_V \frac{\partial\psi}{\partial t}(\mathbf{x}+\boldsymbol{\xi}, t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)\, dV_\xi. \qquad [26]$$

Application of the chain rule to the R.H.S. of [26] yields

$$\left\langle\frac{\partial\psi_\alpha}{\partial t}\right\rangle = \frac{1}{V}\int_V \frac{\partial}{\partial t}[(\psi\mathbf{x}+\boldsymbol{\xi}, t)\gamma_\alpha(\mathbf{x}+\boldsymbol{\xi}, t)]\, dV_\xi - \frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi}, t)\frac{\partial\gamma_\alpha}{\partial t}(\mathbf{x}+\boldsymbol{\xi}, t)\, dV_\xi. \qquad [27]$$

Because V is independent of time, the order of differentiation and integration in the first term on the right side may be reversed and [9] may be invoked to obtain

$$\left\langle\frac{\partial\psi_\alpha}{\partial t}\right\rangle = \frac{\partial}{\partial t}\langle\psi_\alpha\rangle - \frac{1}{V}\int_V \psi(\mathbf{x}+\boldsymbol{\xi}, t)\frac{\partial\gamma_\alpha}{\partial t}(\mathbf{x}+\boldsymbol{\xi}, t)\, dV_\xi. \qquad [28]$$

If the α-phase is deforming, γ_α will be a function of time and the last term in [28] may be non-zero. The total derivative of γ_α with respect to time is

$$\frac{d\gamma_\alpha}{dt} = \frac{\partial\gamma_\alpha}{\partial t} + \frac{dx_1}{dt}\frac{\partial\gamma_\alpha}{\partial x_1} + \frac{dx_2}{dt}\frac{\partial\gamma_\alpha}{\partial x_2} + \frac{dx_3}{dt}\frac{\partial\gamma_\alpha}{\partial x_3}. \qquad [29]$$

Recall that $(\partial\gamma_\alpha/\partial x_1)$, $(\partial\gamma_\alpha/\partial x_2)$, and $(\partial\gamma_\alpha/\partial x_3)$ will be non-zero only on the $A_{\alpha\beta}$ interface. If

(dx_1/dt), (dx_2/dt), and (dx_3/dt) are chosen to be the velocity components of the interface, the total derivative becomes a substantial derivative that moves with the interface. Because an observer riding on the interfacial boundary will see no change, this derivative is zero, or

$$\frac{d\gamma_\alpha}{dt} = 0 = \frac{\partial \gamma_\alpha}{\partial t} + \mathbf{w} \cdot \nabla \gamma_\alpha, \tag{30}$$

where $\mathbf{w}$ is the velocity of the phase interface.

Thus

$$\frac{\partial \gamma_\alpha}{\partial t} = -\mathbf{w} \cdot \nabla \gamma_\alpha, \tag{31}$$

and substitution of this equality into [28] yields

$$\left\langle \frac{\partial \psi_\alpha}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle \psi_\alpha \rangle + \frac{1}{V} \int_V \psi(\mathbf{x} + \boldsymbol{\xi}, t) \mathbf{w}(\mathbf{x} + \boldsymbol{\xi}, t) \cdot \nabla \gamma_\alpha(\mathbf{x} + \boldsymbol{\xi}, t) \, dV_\xi. \tag{32}$$

Identity [8] relates $\nabla \gamma_\alpha$ to the δ-function and [19] indicates that the last term in [32] can be changed to an integral over the α–β phase interface. The final form of this equation is

$$\left\langle \frac{\partial \psi_\alpha}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle \psi_\alpha \rangle - \frac{1}{V} \int_{A_{\alpha\beta}} \psi_\alpha \mathbf{w} \cdot \mathbf{n}_\alpha \, dA, \tag{33}$$

which is a relationship between the spatial average of a time derivative and the time derivative of a spatial average.

CONCLUSION

Use of a distribution function defined to have a value of unity in a phase of interest and zero in all other phases has been shown to be very useful in proving theorems for local volume averaging. The spatial derivative of the distribution function defined in this manner is a three-dimensional form of the Dirac delta function. It has been demonstrated that this property of the distribution function is the source of the surface integrals over phase interfaces that arise in the relationship between averages of derivatives and derivatives of averages.

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